H^p-spaces on compact nilmanifolds

By ADAM KORÁNYI in New York (N.Y., U.S.A.)

To Professor Béla Szőkefalvi-Nagy on his 60th birthday

§ 1. Introduction

The purpose of this paper is to generalize to several complex variables the following classical facts. Let D be the upper halfplane, $\operatorname{Im} z > 0$. In analogy to the usual Hardy spaces $H^p(D)$ one can define the spaces $H^p(\mathbf{Z}\diagdown D)$ of periodic holomorphic functions (with period 1) on D. Adjoining a point at infinity and introducing the new variable $z' = e^{2\pi i z}$, these spaces are transformed to the H^p-spaces of the unit disc; their theory can however be studied without using this transformation. It is immediate that $\{e^{2\pi i m z}\}_{m=0}^{\infty}$ is a complete orthonormal system in $H^2(\mathbf{Z}\diagdown D)$, hence the Cauchy—Szegő kernel (the reproducing kernel of $H^2(\mathbf{Z}\diagdown D)$) is

$$\sum_{m=0}^{\infty} e^{2\pi i m(z-\bar{w})} = \tfrac{1}{2} + \frac{i}{2} \operatorname{cotg} \pi (z - \bar{w}). \tag{1.1}$$

This is also equal to

$$\tfrac{1}{2} + \frac{i}{2\pi}\left[\frac{1}{z-\bar{w}} + \sum_{n \neq 0}\left(\frac{1}{z+n-\bar{w}} - \frac{1}{n}\right)\right] \tag{1.2}$$

which can be regarded as a periodicization of $i(2\pi)^{-1}(z-\bar{w})^{-1}$, the Cauchy—Szegő kernel of D. As z and w approach the real line, (1. 2) becomes a periodicized Hilbert transform kernel, to which the theory of periodic singular integrals [3] applies. In this way, one gets a proof of the theorem of M. Riesz on the conjugate function.

In the present paper, D will be a generalized halfplane analytically equivalent with the unit ball in $\mathbf{C}^n$ $(n > 1)$. For simplicity of notation, we will assume $n = 2$, but everything extends to the general case. The boundary of D can be identified with a certain nilpotent Lie group $\mathfrak{N}$, often referred to as a Heisenberg group, and the role of $\mathbf{Z}$ is taken by a discrete co-compact subgroup Γ of $\mathfrak{N}$. In the case $n > 1$, $\Gamma\diagdown D$ is not contained in any Stein manifold, as one sees easily by the boundedness of holomorphic functions at infinity (§ 4) and by topological reasons. We prove the analogue of the identity of (1. 1) and (1. 2) by Fourier transform methods ((5. 3),

(5. 6), Theorem 5. 2). In § 2, we develop in a slightly more general setting the necessary theorems about periodic singular integrals on nilpotent groups. In § 6, we prove the generalized M. Riesz theorem.

Along the way, in § 4, we also discuss the position of $H^2(\Gamma\backslash D)$ in relation to the harmonic analysis of the regular representation on $L^2(\Gamma\backslash\mathfrak{N})$ and we point out that the transformation formula for θ-functions is, up to a constant factor of modulus one, a consequence of our results. In § 7, we make some observations about the Bergman kernel of $\Gamma\backslash D$.

Independently and in a different context, H. Rossi has also considered the manifold $\Gamma\backslash D$ and also constructed the orthogonal system (4. 3). I wish to express to him my thanks for several useful conversations on this subject.

§ 2. Singular integrals on compact nilmanifolds

Let G be a connected, simply connected nilpotent Lie group, e its identity element. Let $\{a(t)\}$ be a multiplicatively written one-parameter group of automorphisms of G such that

$$\lim_{t\to 0} a(t)\, g = e \tag{2.1}$$

for all $g\in G$, and such that the induced automorphisms $a_*(t)$ of the Lie algebra are diagonalizable.

Let dg be a fixed Haar measure on G. Then there exists $q>0$ such that

$$d(a(t)g) = t^q\, dg \tag{2.2}$$

for all $t>0$.

Given any real s, a function f defined on $G-\{e\}$ is said to be *homogeneous of degree* s if, for all $t>0$, $g\in G$,

$$f(a(t)g) = t^s f(g).$$

Let $g\mapsto |g|$ be a strictly positive, continuously differentiable function on $G-\{e\}$, homogeneous of degree 1, and such that $|g^{-1}|=|g|$. This function is then necessarily a *gauge* in the sense of [7], i.e.

$$|gh| \leqq \varkappa(|g|+|h|) \tag{2.3}$$

for all $g, h\in G$ with some universal $\varkappa$, and the sets

$$B(r) = \{g\in G \,|\, |g| < r\}$$

which form a base of neighborhoods of e, have measure $m(B(r))$ proportional to r^q.

In [7], the largest eigenvalue of the infinitesimal generator of $\{a_*(t)\}$ was denoted by α. As explained on p. 604 of [7], it is no restriction of generality to assume $\alpha=1$.

We will assume this in this section; our results (Theorems 2. 1 and 2. 2) are, of course, valid without change for any value of α.

Before stating our first theorem, we recall the special cases of Lemmas 5. 1. and 5. 2. in [7] which we will need repeatedly:

Let f be homogeneous of degree s and continuously differentiable on $G-\{e\}$. Then, for all $0<a<b$,

$$\int_{a<|g|<b} f(g)\,dg = \begin{cases} C(b^{q+s}-a^{q+s}) & \text{if } s\neq -q \\ C(\log b-\log a) & \text{if } s=-q \end{cases} \tag{2.4}$$

with some constant C. Furthermore, there exist numbers $M, N\geqq 1$ such that,

$$\left.\begin{matrix} |f(gh)-f(g)| \\ |f(hg)-f(g)| \end{matrix}\right\} < M|h|\,|g|^{s-1} \tag{2.5}$$

whenever $N|h|<|g|$.

Theorem 2. 1. *Let $k:G-\{e\}\to\mathbf{R}$ be continuously differentiable, homogeneous of degree $-q$, and such that*

$$\int_{a<|g|<b} k(g)\,dg = 0$$

for some (hence all) $0<a<b$. Let $k(e)=0$. Let Γ be a discrete subgroup of G. Then, for g, h in any compact subset of G,

$$k^*(g,h) = \sum_{\gamma\in\Gamma}[k(h^{-1}\gamma g)-k(\gamma)] \tag{2.6}$$

converges normally after the omission of finitely many terms. For any $\gamma,\delta\in\Gamma$.

$$k^*(\gamma g,\delta h) = k^*(g,h). \tag{2.7}$$

Proof. Let g, h be in a compact subset; then $|g|$, $|h|\leqq C$ with the some constant C. For $|g|\leqq C$ and $|\gamma|$ large we have

$$|\gamma g| \geqq \tfrac{1}{2}|\gamma|. \tag{2.8}$$

In fact, this follows since the homogeneity of the gauge and (2. 5) imply $\big||\gamma g|-|\gamma|\big| < M|g|$ whenever $N|g|<|\gamma|$.

By (2. 8) and (2. 5), it follows that

$$|k(h^{-1}\gamma g)-k(\gamma)| \leqq C'|\gamma|^{-q-1}$$

for $|h|$, $|g|\leqq C$ and $|\gamma|$ large. The first assertion of the theorem follows if we show

$$\sum_{|\gamma|>R}|\gamma|^{-q-1}<\infty$$

for some R.

Let $r>0$ be such that the sets $\gamma B(r)$ $(\gamma\in\Gamma)$ are all disjoint. Now, for $|g|<r$ and $|\gamma|>R$, with an appropriate R, we have (2. 8), and hence

$$\sum_{|\gamma|>R}|\gamma|^{-q-1}\leqq\frac{2^{q+1}}{m(B(r))}\sum_{|\gamma|>R}\int_{B(r)}|\gamma g|^{-q-1}\,dg\leqq\frac{2^{q+1}}{m(B(r))}\int_{|g|>R/2}|g|^{-q-1}\,dg$$

which is finite by (2. 4).

To prove the second assertion, we show that $k^*(\delta g, h)=k^*(g, h)$ for any fixed $\delta\in\Gamma$. Since the same method also proves $k^*(g, \delta h)=k^*(g, h)$, this will finish the proof.

By absolute convergence of (2. 6), it suffices to show that

$$\sum_{\gamma\in\Gamma}[k(h^{-1}\gamma\delta g)-k(h^{-1}\gamma g)]=0. \tag{2.9}$$

We note that, for large R, $|\gamma|<R-M|\delta|$ implies

$$|\gamma\delta|<R. \tag{2.10}$$

This follows from (2. 5) when $|\gamma|>N|\delta|$ and from (2. 3) when $|\gamma|\leqq N|\delta|$.

Consequently,

$$\sum_{|\gamma|<R}[k(h^{-1}\gamma\delta g)-k(h^{-1}\gamma g)] \tag{2.11}$$

is majorized by

$$\sum_{R-M|\delta|<|\gamma|<R+M|\delta|}|k(h^{-1}\gamma g)|.$$

By homogeneity, $|k(g)|\leqq C|g|^{-q}$. We use (2. 8) as in the proof of the first assertion, and then use (2. 10) which is true for any fixed δ (not necessarily in Γ) to obtain the majorization of (2. 11) by

$$\frac{2^{q+1}C}{m(B(r))}\int_{R-c<|g|<R+c}|g|^{-q}dg$$

where $c=M|\delta|+Mr$. By (2. 4) this integral tends to 0 as $R\to\infty$, finishing the proof.

From now on let Γ be a discrete subgroup such that $\Gamma\backslash G$ is compact. As usual, we will identify functions on $\Gamma\backslash G$ with functions f on G such that $f(\gamma g)=f(g)$ for all $\gamma\in\Gamma$, $g\in G$. We choose a fundamental domain Ω for the action of Γ on G such that $e\in\Omega$; as it is shown in [1] (p. 54), Ω can be chosen as the exponential image of an interval in the Lie algebra. There is a right G-invariant measure on $\Gamma\backslash G$ given by

$$\int_{\Gamma\backslash G}f=\int_{\Omega}f(g)\,dg.$$

We denote by $L^p(\Gamma\backslash G)$ the usual L^p-spaces with respect to this measure.

For $g, h\in G$, we define

$$\varrho(g, h)=\inf_{\gamma\in\Gamma}|h^{-1}\gamma g|.$$

It is clear from (2. 3) that, for any $g, h, l \in G$,

$$\varrho(g, l) \leqq \varkappa(\varrho(g, h)+\varrho(h, l)).$$

Also, $\varrho(\gamma g, \delta h)=\varrho(g, h)$ for $\gamma, \delta \in \Gamma$, so ϱ can be regarded as a pseudodistance on $\Gamma \diagdown G$.

Given k and k^* as in Theorem 2. 1, we define, for small $\varepsilon>0$,

$$k^{*, \varepsilon}(g, h) = \begin{cases} k^*(g, h) & \text{if } \varrho(g, h) \geqq \varepsilon \\ 0 & \text{if } \varrho(g, h) < \varepsilon \end{cases}$$

This is a bounded measurable function on $\Gamma \diagdown G$, therefore the operator A^ε defined on $L^p(\Gamma \diagdown G)$ $(1<p<\infty)$ by

$$(A^\varepsilon f)(g) = \int_{\Gamma \diagdown G} f(h) k^{*, \varepsilon}(g, h)\, dh$$

is bounded.

As in [7], we also define k^ε on G by

$$k^\varepsilon(g) = \begin{cases} k(g) & \text{if } |g| \geqq \varepsilon \\ 0 & \text{if } |g| < \varepsilon \end{cases}$$

Theorem 2. 2. *Let* $1<p<\infty$. *For all* $f \in L^p(\Gamma \diagdown G)$, $Af = \lim_{\varepsilon \to 0} A^\varepsilon f$ *exists in* $L^p(\Gamma \diagdown G)$, *and* A *is a bounded operator on* $L^p(\Gamma \diagdown G)$.

Proof. Let

$$E = \{(g, h) \mid g \in \Omega, h \in \Omega g\}. \tag{2. 12}$$

Then $k^*(g, h)-k(h^{-1}g)$ is bounded on E; this follows from the easily proven fact that, for small $\varepsilon>0$, $(g, h) \in E$ implies $|h^{-1}\gamma g|>\varepsilon$ for all $e \neq \gamma \in \Gamma$.

Define, for, for $g \in \Omega$,

$$(Tf)(g) = \int_{\Omega g} f(h)[k^*(g, h)-k(h^{-1}g)]\, dh$$

$$(T^\varepsilon f)(g) = \int_{\Omega g} f(h)[k^{*, \varepsilon}(g, h)-k^\varepsilon(h^{-1}g)]\, dh.$$

Since, for fixed g, Ωg is also a fundamental domain for Γ, it is clear by classical arguments that T, T^ε are bounded operators $L^p(\Gamma \diagdown G) \to L^p(\Omega)$ and $\lim_{\varepsilon \to 0} T^\varepsilon = T$ in the strong operator topology.

Let $A_1^\varepsilon = A^\varepsilon - T^\varepsilon$ regarded as an operator $L^p(\Gamma \diagdown G) \to L^p(\Omega)$. To prove the theorem it is enough to show that $\lim_{\varepsilon \to 0} A_1^\varepsilon$ exists strongly. For this, in turn, it is enough to show that if $\varepsilon, \eta \to 0$, then $A_1^{\varepsilon,\eta} f = A_1^\varepsilon f - A_1^\eta f \to 0$ in $L^p(\Omega)$ for all $f \in L^p(\Gamma \diagdown G)$. Note that we have, for $g \in \Omega$,

$$(A_1^\varepsilon f)(g) = \int_{\Omega g} f(h) k^\varepsilon (h^{-1})\, dh.$$

We define the function f_1 on G by

$$f_1(g) = \begin{cases} f(g) & \text{if} \quad g \in \Omega^2, \\ 0 & \text{if} \quad g \notin \Omega^2. \end{cases}$$

By compactness, Ω^2 is covered by finitely many translates of Ω; hence $\|f_1\|_{L^p(G)} \leqq M \|f\|_{L^p(\Gamma \diagdown G)}$ with some constant M.

For functions on G, let B^ε denote the operator of convolution by k^ε. By [7] (Theorem 5. 1), $\lim\limits_{\varepsilon \to 0} B^\varepsilon$ exists strongly. So $B^{\varepsilon,\eta} f_1 = B^\varepsilon f_1 - B^\eta f_1 \to 0$ in $L^p(G)$ as $\varepsilon, \eta \to 0$. Hence, $B^{\varepsilon,\eta} f_1 \to 0$ also as an element of $L^p(\Omega)$.

Now, for $g \in \Omega$, we have, by a change of variable,

$$A_1^{\varepsilon,\eta} f(g) - B^{\varepsilon,\eta} f(g) = \int_{G - g^{-1}\Omega g} f_1(gl) [k^\varepsilon (l^{-1}) - k^\eta (l^{-1})]\, dl.$$

By compactness, there exists $r_0 > 0$ such that $B(r_0) \subset g^{-1} \Omega g$ for all $g \in \Omega$. Thus, the last integral is 0 for $\varepsilon, \eta < r_0$, finishing the proof.

Remark. It is possible to show that the singular integral in the sense of [7] converges also a.e., not only in $L^p(G)$. Using this, it is easy to show by the method of [3] that our $\lim\limits_{\varepsilon \to 0} A^\varepsilon f(g)$ also exists for a.a. g. With this fact available, the proof of Theorem 2. 2. can be slightly simplified along the lines of [3].

§ 3. H^p-spaces

In the following, we consider the domain D of [7, § 6], but only in the special case of two variables. The reason for this restriction is that all significant features of our problem already appear in this special case. Everything that follows is trivially generalizable to n variables, the material of the present section even to any generalized halfplane.

So, let

$$D = \{(z_1, z_2) \in \mathbf{C}^2 \mid \operatorname{Im} z_1 - \tfrac{1}{2} |z_2|^2 > 0\}$$

and let B be its boundary. Let $\mathfrak{N}$ be the subgroup of the group of holomorphic automorphisms of D which as a set equals $\mathbf{R} \times \mathbf{C}$, an element (ξ, ζ) acting by

$$z_1 \mapsto z_1 + \xi + i\bar{\zeta} z_2 + \frac{i}{2} |\zeta|^2$$
$$z_2 \mapsto z_2 + \zeta$$

$\mathfrak{N}$ is simply transitive on B, which gives a natural identification $\mathfrak{N}\ni(\xi,\zeta)=g\mapsto$ $\mapsto g\cdot 0=\left(\xi+\frac{i}{2}|\zeta|^2,\zeta\right)\in B$. Lebesgue measure on $\mathbf{R}\times\mathbf{C}$ corresponds to a Haar measure on $\mathfrak{N}$. The group $\{a(t)\}$ of automorphisms given by

$$a(t)(\xi,\zeta)=(t\xi,t^{1/2}\zeta)$$

has the properties (2. 1), (2. 2) will $q=2$ and we have a smooth homogeneous gauge on $\mathfrak{N}$ given by

$$|(\xi,\zeta)|=(\xi^2+\tfrac{1}{4}|\zeta|^4)^{1/2}.$$

Let Γ be a discrete subgroup of $\mathfrak{N}$ such that $\Gamma\diagdown\mathfrak{N}$ is compact. Then, $\Gamma\diagdown D$ is a complex manifold with boundary $\Gamma\diagdown\mathfrak{N}$ (since $\mathfrak{N}$ is identified with B). Again, we identity functions on $\Gamma\diagdown D$ with functions f on D such that $f\circ\gamma=f$ for all $\gamma\in\Gamma$.

As in [6], for $t>0$, we denote $f_t(z_1,z_2)=f(z_1+it,z_2)$ and $\tilde{f}_t(g)=f_t(g\cdot 0)$. If $f\circ\gamma=f$, the same is true for f_t and $\tilde{f}_t$. So, we may define $H^p(\Gamma\diagdown D)$ as the space of all holomorphic functions f on $\Gamma\diagdown D$ for which

$$\|f\|_{H^p(\Gamma\diagdown D)}=\sup_{t>0}\|\tilde{f}_t\|_{L^p(\Gamma\diagdown\mathfrak{N})}<\infty.$$

Theorem 3. 1. *Let* $1<p<\infty$. *If* $f\in H^p(\Gamma\diagdown D)$, *then* $\tilde{f}=\lim_{t\to 0}\tilde{f}_t$ *exists in* $L^p(\Gamma\diagdown\mathfrak{N})$, *and* $f\mapsto\tilde{f}$ *is an isometric imbedding.*

Proof. As known from [6], when f is bounded, continuous on $\bar{D}$ and holomorphic on D, then it has a Poisson integral representation

$$\tilde{f}_t(g)=\int_{\mathfrak{N}}\tilde{P}^t(h^{-1}g)f(h\cdot 0)\,dh \tag{3.1}$$

where, for $g=(\xi,\zeta)$,

$$\tilde{P}^t(g)=\frac{1}{2\pi^2}\frac{2}{[\xi^2+(t+\frac{1}{2}|\zeta|^2)^2]^2}. \tag{3.2}$$

It is easy to see that for h, g in a compact set and for large $|\gamma|$,

$$|\tilde{P}^t(h^{-1}\gamma g)|\leqq C\frac{t^2}{|\gamma|^4}. \tag{3.3}$$

This shows the normal convergence of the series

$$P_t^*(g,h)=\sum_{\gamma\in\Gamma}\tilde{P}^t(h^{-1}\gamma g)$$

and therefore also that, for f bounded continuous on $\Gamma\diagdown\bar{D}$ and holomorphic on $\Gamma\diagdown D$, we have

$$\tilde{f}_t(g)=\int_{\Gamma\diagdown\mathfrak{N}}P_t^*(g,h)f(h\cdot 0)\,dh. \tag{3.4}$$

Now, let $f\in H^p(\Gamma\backslash D)$. For each fixed $t_0>0$, f_{t_0} has a representation (3. 4). (The boundedness of f_{t_0} follows from the usual subharmonicity argument for any generalized halfplane D; in our special case, it is obvious from the fact that holomorphic functions on $\Gamma\backslash D$ are automatically bounded at infinity, cf. § 4). The standard weak compactness argument shows now that f is the Poisson integral of some function in $L^p(\Gamma\backslash\mathfrak{N})$.

Next, we show that whenever f is the Poisson integral of some $\varphi\in L^p(\Gamma\backslash\mathfrak{N})$ we have $\lim_{t\to 0}\tilde{f}_t=\varphi$ in $L^p(\Gamma\backslash\mathfrak{N})$. In fact, using Jensen's inequality [12, Vol. I, p. 24].

$$\|\varphi-\tilde{f}_t\|^p\leqq\int_{\Omega}dg\int_{\Omega g}P_t^*(g,h)|\,\varphi(g)-\varphi(h)|^p\,dh\leqq$$

$$\leqq\int\int_{\substack{(g,h)\in E\\|h^{-1}g|<\eta}}\tilde{P}^t(h^{-1}g)|\varphi(g)-\varphi(h)|^p\,dg\,dh+$$

$$+\int\int_{\substack{(g,h)\in E\\|h^{-1}g|<\eta}}[P_t^*(g,h)-\tilde{P}^t(h^{-1}g)]\,|\varphi(g)-\varphi(h)|^p\,dg\,dh+$$

$$+\int\int_{\substack{(g,h)\in E\\|h^{-1}g|>\eta}}P_t^*(g,h)|\,\varphi(g)-\varphi(h)|^p\,dg\,dh.$$

For η small enough, the first integral is small by the results of [6]; the second is small since $P_t^*(g,h)-\tilde{P}^t(h^{-1}g)$ is bounded on E. Once η is chosen, (3. 3) shows that the third integral can be made small by choosing t small enough.

We see now that, for $f\in H^p(\Gamma\backslash D)$, $\tilde{f}=\lim_{t\to 0}\tilde{f}_t$ exists in $L^p(\Gamma\backslash\mathfrak{N})$ and f is the Poisson integral of $\tilde{f}$. The latter fact and Jensen's inequality show $\|\tilde{f}_t\|\leqq\|\tilde{f}\|$ which implies $\|\tilde{f}\|_{L^p(\Gamma\backslash\mathfrak{N})}=\|f\|_{H^p(\Gamma\backslash D)}$, finishing the proof.

Corollary. *$H^2(\Gamma\backslash D)$ is a Hilbert space.*

§ 4. An orthonormal system in $H^2(\Gamma\backslash D)$

In this section, we specify Γ. Let k be a positive integer and τ a complex number such that $\operatorname{Im}\tau>0$. Let

$$\Gamma_\tau^k=\left\{(a\operatorname{Im}\tau,b+c\tau)\middle|\ b,c,\frac{k}{2}(a+bc)\in\mathbf{Z}\right\}.$$

Using some arguments of Brezin [2, p. 614], it is not hard to show that we get all possible complex manifolds $\Gamma\backslash D$ up to isomorphism by taking $\Gamma=\Gamma_\tau^k$. Different values of τ give isomorphic complex manifolds if they are equivalent under the modular group. (It is well known [1] that the nilmanifolds $\Gamma_\tau^k\backslash\mathfrak{N}$ are isomorphic for any two values of τ.)

The inequalities

$$|\xi| < \frac{\operatorname{Im}\tau}{k}, \quad |\operatorname{Re}\zeta| < \tfrac{1}{2}, \quad |\operatorname{Im}\zeta| < \frac{\operatorname{Im}\tau}{2}$$

determine a fundamental domain Ω for Γ_τ^k in $\mathfrak{N}$.

It is immediate [9, p. 137] that every function f holomorphic on $\Gamma_\tau^k \diagdown D$ has a Fourier expansion

$$f(z_1, z_2) = \sum_{m \equiv 0 \pmod k} e^{(\operatorname{Im}\tau)^{-1} m\pi i z_1} \psi_m(z_2)$$

with ψ_m holomorphic and satisfying

$$\psi_m(z+\zeta) = \psi_m(z) e^{(\operatorname{Im}\tau)^{-1}\pi m(\bar\zeta z + \frac{1}{2}|\zeta|^2)} \tag{4.1}$$

for all $\zeta = b + c\tau$ $(b, c \in \mathbf{Z})$. Defining χ_m by

$$\chi_m(z) = e^{-\frac{\pi}{2}(\operatorname{Im}\tau)^{-1} m z^2} \psi_m(z)$$

(4. 1) is seen to be equivalent with the pair of equations

$$\chi_m(z+1) = \chi_m(z), \quad \chi_m(z+\tau) = e^{-m\pi i(\tau + 2z)} \chi_m(z).$$

These are the standard functional equations of θ-functions. They have holomorphic solutions only for $m \geqq 0$. As first observed by PJATECKIĬ-ŠAPIRO [9, p. 140], from this it is immediate that every holomorphic f is bounded as $\operatorname{Im} z_1 \to \infty$. For $m=0$, the only solutions are the constants; for $m>0$, a basis of the space of solutions is given by

$$\chi_{ml}(z) = \sum_{j \equiv l \pmod m} e^{\frac{\pi i \tau}{m} j^2} e^{2\pi i j z} \tag{4.2}$$

$(0 \leqq l \leqq m-1)$, cf. [5], [10] where the several variable case is also handled, showing how to extend our results to n-dimensional D.

By Theorem 3. 1., the computation of inner products in H^2 reduces to simple integrations on Ω. An easy computation shows that

$$\begin{cases} \varphi_{00} \equiv 1, \\ \varphi_{ml}(z_1, z_2) = e^{(\operatorname{Im}\tau)^{-1}\pi m(i z_1 + \frac{1}{2} z_2^2)} \chi_{ml}(z_2) \end{cases} \tag{4.3}$$

$(0 \leqq l < m,\ m \equiv 0 \pmod k)$ is an orthogonal basis of $H^2(\Gamma_\tau^k \diagdown D)$. An application of the Parseval identity gives the norms:

$$\|\varphi_{ml}\|^2_{H^2(\Gamma\diagdown D)} = \begin{cases} \dfrac{2(\operatorname{Im}\tau)^2}{k} & \text{if} \quad (m, l) = (0, 0) \\ \dfrac{2^{1/2}(\operatorname{Im}\tau)^{3/2}}{k m^{1/2}} & \text{if} \quad (m, l) \neq (0, 0). \end{cases} \tag{4.4}$$

$L^2(\Gamma_\tau^k \backslash \mathfrak{N})$ carries a unitary representation of $\mathfrak{N}$, the "right regular representation", whose harmonic analysis is well known [8], [2]. We wish to elucidate the position of $H^2(\Gamma_\tau^k \backslash D)$ and of the system (4. 3) in this context. We have [8]

$$L^2(\Gamma_\tau^k \backslash \mathfrak{N}) = \bigoplus_{m \equiv 0 \ (\mathrm{mod}\ k)} L(m)$$

where $L(0)$ consists of constants and, for $m \neq 0$, $L(m)$ is the sum of $|m|$ copies of the irreducible representation of $\mathfrak{N}$ determined by the character m of the center.

Let $\{X, Y, Z\}$ be the basis of the Lie algebra such that $\exp tX = (t, 0)$, $\exp Y = (it, 0)$, $\exp tZ = (0, t)$. It is known [4] that for an irreducible representation π of $\mathfrak{N}$, the space of solutions f of $d\pi(X+iY)f = 0$ (sometimes called the vacuum subspace) is one-dimensional.

In the present case, it is clear that $\tilde{\varphi}_{ml} \in L(m)$, also we have $\left.\frac{\partial \varphi_{ml}}{\partial \bar{z}_2}\right|_0 = 0$. An easy computation gives $\left.\frac{\partial f}{\partial \bar{z}_2}\right|_0 = dR(X+iY)\tilde{f}|_e$ for any f differentiable on $\bar{D}$. Since X, Y are left invariant and since $\mathfrak{N}$ acts holomorphically, it follows that $dR(X+iY)\tilde{\varphi}_{ml} = 0$.

Thus, the space spanned by $\{\tilde{\varphi}_{m,0}, \ldots, \tilde{\varphi}_{m,m-1}\}$ $(m > 0)$ is exactly the "vacuum subspace" of $L(m)$. If we denote by L_{ml} the space spanned by all right translates of $\tilde{\varphi}_{ml}$, we have

$$L(m) = \bigoplus_{l=0}^{m-1} L_{ml}$$

a decomposition of $L(m)$ into irreducible subspaces. By the method of BREZIN [2] one can now construct a concrete orthogonal basis of each L_{ml} $(m > 0)$ which contains $\tilde{\varphi}_{ml}$ as its first element.

As a curiosity, we mention also the following. If $\tau' = -\frac{1}{\tau}$, then $\Gamma_\tau^1 \backslash D$ and $\Gamma_{\tau'}^1 \backslash D$ are isomorphic as complex manifolds under the map $\iota : (z_1, z_2) \mapsto (|\tau|^2 z_1, \tau z_2)$ and the map $f \mapsto |\tau|^2 f \circ \iota$ is an isomorphism of the corresponding H^2-spaces. Since $H^2(\Gamma_\tau^1 \backslash D) \cap L(1)$ is spanned by $\tilde{\varphi}_{10}$, we have therefore $c|\tau|^2 \|\varphi_{10}\|^{-1} \varphi_{10} \circ \iota = \|\varphi'_{10}\|^{-1} \varphi_{10}$, where φ'_{10} is constructed from τ' the way φ_{10} is from τ, and $|c| = 1$. After a short computation this reduces to

$$c|\tau|^{-1/2} e^{\pi \frac{z^2}{i\tau}} \chi'_{10}\left(\frac{z}{\tau}\right) = \chi_{10}(z)$$

where χ'_{10} is given by (4. 2) with τ' instead of τ. This equation, except for the exact value of the argument of c, is the fundamental transformation formula of θ-functions.

§ 5. The Szegő kernel of $\Gamma\backslash D$

In this section, we fix one of the groups Γ_τ^k and denote it briefly by Γ. The Szegő kernel S of D is given by [6]

(5. 1) $$S(z, w) = \frac{1}{2\pi^2}[i(z_1-\bar{w}_1)-z_2\bar{w}_2]^{-2}$$

We define

(5. 2) $$c^\Gamma = \frac{k}{4(\operatorname{Im}\tau)^2}+\lim_{r\to\infty}\sum_{0\neq|\gamma|<r} S(\gamma\cdot 0, 0).$$

It will be seen in the proof of Theorem 3. 1 that this limit exists. Further, we define

(5. 3) $$S^*(z, w) = c^\Gamma + S(z, w) + \sum_{e\neq\gamma}[S(\gamma z, w)-S(\gamma\cdot 0, 0)].$$

Lemma 5. 1. *For z, w in any bounded subset of $\mathbf{C}^2\times\mathbf{C}^2$, the sum in* (5. 3) *converges normally after the omission of finitely many terms.*

Proof. From (5. 1) a simple explicit computation gives

$$|S(\gamma z, w)-S(\gamma\cdot 0, 0)| \leqq C|\gamma|^{-5/2}$$

for $|\gamma|$ large. Therefore, except for finitely many terms, the series is majorized by $\sum|\gamma|^{-5/2}$ which converges by (2. 4) and the argument of Theorem 2. 1.

Theorem 5. 2. *S^* is the Szegő kernel of $\Gamma\backslash D$, i.e. the reproducing kernel of the Hilbert space $H^2(\Gamma\backslash D)$.*

Proof. By the same argument as in Theorem 2. 1 one shows that $S^*(\gamma z, \delta w) = S(z, w)$ for all $\gamma, \delta\in\Gamma$. Thus S^* can be regarded as a function on $\Gamma\backslash D\times\Gamma\backslash D$.

For fixed $w\in D$, we introduce the usual notation $S_w^*(z) = S^*(z, w)$. S_w^* is a holomorphic function on $\bar{D}$ (and on $\Gamma\backslash\bar{D}$); this follows since, by Lemma 5. 1, S_w^* is meromorphic everywhere and since none of the terms in (5. 3) have any pole on $\bar{D}$. It follows now that S_w^* is bounded at infinity (cf. § 4), and hence that $S_w^*\in H^2(\Gamma\backslash D)$.

From the Poisson integral representation (Theorem 3. 1), it follows that point evaluations are continuous functionals on $H^2(\Gamma\backslash D)$. Therefore to prove the Theorem, it is enough to prove that $(f, S_w^*) = f(w)$ for all w and for a system of functions that span $H^2(\Gamma\backslash D)$. We will show this for the orthogonal basis $\{\varphi_{ml}\}$.

First, we have

$$(\varphi_{00}, S_w^*) = \int_\Omega S^*(g\cdot 0, w)\,dg =$$

$$= c^\Gamma\frac{2(\operatorname{Im}\tau)^2}{k}+\lim_{r\to\infty}\left[\int_{|g|<r} S(g\cdot 0, w)\,dg - \frac{2(\operatorname{Im}\tau)^2}{k}\sum_{0\neq|\gamma|<r} S(\gamma\cdot 0, 0)\right].$$

By [7, Lemma 6. 2], the limit of the integral on the right is 1/2. This shows that the limit in (5. 2) exists, and also that $(\varphi_{00}, S_w^*) = 1 = \varphi_{00}(w)$.

Now, let $(m, l)\neq(0, 0)$. To compute (φ_{ml}, S_w^*) we note that by normal convergence of (5. 3) the series can be rearranged in any way and integrations can be performed term by term. Since clearly we have $\int\limits_{\Omega}\varphi_{ml}=0$, it follows that

$$(5.4)\qquad (\varphi_{m,l}, S_w^*) = \sum_{\gamma\in\Gamma}\int\limits_{\Omega} S(w, \gamma u)\varphi_{ml}(u)\,d\beta(u)$$

where, as in [6], we denote $u = \left(x_1+\frac{i}{2}|z_2|^2, z_2\right)$, $z_2 = x_2+iy_2$, and $d\beta(u)= =dx_1\,dx_2\,dy_2$.

We have

$$S(w, z) = 2\int\limits_0^\infty e^{2\pi\lambda[i(z_1-\overline{w}_1)-z_2\overline{w}_2]}\lambda\,d\lambda$$

which shows that $S(w, u)$ as a function of x_1 is the Fourier transform of a continuous L^2-function. By [11, Theorem 58], the Fourier—Plancherel inversion formula is applicable pointwise (not only a.e.); thus, using (4. 3) and (4. 1),

$$\lim_{N\to\infty}\sum_{|\xi|\le N}\int\limits_{-\frac{\operatorname{Im}\tau}{k}}^{\frac{\operatorname{Im}\tau}{k}} S\big(w, (\xi,\zeta)\cdot u\big)\varphi_{ml}(u)\,dx_1 =$$

$$= \frac{m}{\operatorname{Im}\tau}\, e^{(\operatorname{Im}\tau)^{-1}\pi m[iw_1-|z_2+\zeta|^2+w_2(\bar z_2+\bar\zeta)+\frac{1}{2}(z_2+\zeta)^2]}\chi_{ml}(z_2+\zeta).$$

To find (5. 4), we have to integrate this with respect to x_2 and y_2, and sum over ζ. This gives

$$(5.5)\qquad \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{m}{\operatorname{Im}\tau}\, e^{(\operatorname{Im}\tau)^{-1}\pi m[iw_1-|z_2|^2+\bar z_2 w_2+\frac{1}{2}z_2^2]}\chi_{ml}(z_2)\,dx_2\,dy_2$$

To justify this step, it is enough to show that the integral exists absolutely. This follows by considering the series (4. 2) which defines χ_{ml}; for each term separately, the absolute integral exists and their sum is convergent.

This also shows that (5. 5) can be computed by substituting (4. 2) and integrating term-by-term. This is a direct computation which shows that (5.5) equals $\varphi_{ml}(w)$, finishing the proof.

Remark. From (4. 3), (4. 4) it follows by general principles that

$$(5.6)\qquad S^*(z, w) = \frac{k}{2(\operatorname{Im}\tau)^2}+\frac{k}{2^{1/2}(\operatorname{Im}\tau)^{3/2}}\sum_{\substack{m\equiv 0 \ (\mathrm{mod}\ k)\\ m>0}} m^{1/2}\sum_{l=0}^{m-1}\varphi_{ml}(z)\overline{\varphi_{ml}(w)}$$

§ 6. The generalized M. Riesz theorem

Theorem 5. 2 shows that for $f \in L^2(\Gamma \backslash \mathfrak{N})$

$$Pf(z) = \int_{\Gamma \backslash \mathfrak{N}} S^*(z, g \cdot 0) f(g)\, dg \tag{6. 1}$$

gives the orthogonal projection of f onto $H^2(\Gamma \backslash D)$. We will show that (6. 1) also defines a bounded projection $L^p(\Gamma \backslash \mathfrak{N}) \to H^p(\Gamma \backslash D)$ for all $1 < p < \infty$. In the one-variable case this is an equivalent formulation of the classical M. Riesz theorem on the conjugate function.

As in [7], we denote for $t > 0$, $g = (\xi, \zeta)$.

$$k_t(g) = \frac{1}{2\pi^2} (t + \tfrac{1}{2}|\zeta|^2 - i\xi)^{-2}$$

$$k(g) = \frac{1}{2\pi^2} (\tfrac{1}{2}|\zeta|^2 - i\xi)^{-2}.$$

As shown in [7, § 6], k is a kernel satisfying the conditions of Theorem 2. 2.

We define k^ε, k^*, $k^{*,\varepsilon}$ as in § 2. Furthermore, we define

$$k_t^*(g, h) = c^\Gamma + \sum_{\gamma \in \Gamma} [k_t(h^{-1} \gamma g) - k(\gamma)]$$

Clearly, we have $k_t^*(g, h) = S^*(g \cdot 0, h(it, 0))$, and therefore

$$(Pf)_t^\sim(g) = \int_{\Gamma \backslash \mathfrak{N}} k_t^*(g, h) f(h)\, dh. \tag{6. 2}$$

Lemma 6. 1. *Let $1 < p < \infty$. Then, for all $f \in L^p(\Gamma \backslash \mathfrak{N})$,*

$$\lim_{\varepsilon \to 0} \int_{\Gamma \backslash \mathfrak{N}} f(h) [k_\varepsilon^*(g, h) - k^{*,\varepsilon}(g, h)]\, dh = \tfrac{1}{2} f(g)$$

in the sense of convergence in $L^p(\Gamma \backslash \mathfrak{N})$.

Proof. We give a sketch, omitting much tedious detail. Let E be the set (2. 12) Then, there exists $\varepsilon_0 > 0$ such that the series

$$k_\varepsilon^*(g, h) - k_\varepsilon(h^{-1} g) = \sum_{\gamma \neq e} [k_\varepsilon(h^{-1} \gamma g) - k(\gamma)]$$

is normally convergent for $(g, h) \in E$, $0 < \varepsilon \leqq \varepsilon_0$. This is a slight extension of Lemma 5. 1 and can be verified by some explicit computation. Using this, one shows next that

$$\sum_{\gamma \neq e} [k_\varepsilon(h^{-1} \gamma g) - k^\varepsilon(h^{-1} \gamma g)]$$

is bounded for $(g, h) \in E$, $0 < \varepsilon \leqq \varepsilon_0$, and tends to 0 as $\varepsilon \to 0$. Thus, the integral operator defined on $L^p(\Omega)$ by this kernel tends to 0. This reduces the proof of the lemma to proving that

$$\lim_{\varepsilon\to 0}\int_{\Omega g} f(h)\,[k_\varepsilon(h^{-1}g)-k^\varepsilon(h^{-1}g)]\,dh=\tfrac{1}{2}f(g).$$

Making a change of variable, we have the identity

$$\int_{\Omega g} f(h)\,[k_\varepsilon(h^{-1}g)-k^\varepsilon(h^{-1}g)\,dh=I_1+I_2+I_3,$$

$$I_1=f(g)\int_{g^{-1}\Omega g}[k_\varepsilon(h^{-1})-k^\varepsilon(h^{-1})]\,dh,$$

$$I_2=\int_{\substack{g^{-1}\Omega g\\ |h|>\varepsilon}}[f(gh)-f(h)]\,[k_\varepsilon(h^{-1})-k^\varepsilon(h^{-1})]\,dh,$$

$$I_3=\int_{\substack{g^{-1}\Omega g\\ |h|<\varepsilon}}[f(gh)-f(g)]\,k_\varepsilon(h^{-1})\,dh.$$

By compactness, there exists $r>0$ such that $B(r)\subset g^{-1}\Omega g$ for all $g\in\Omega$. I_1 can be written as the sum of an integral on $B(r)$ and one on its complement. The first tends to $\frac{1}{2}f(g)$ as $\varepsilon\to 0$, by the corollary of [7, Lemma 6. 2]. The second tends to 0 since the integrand tends to 0 uniformly. One shows that I_2, I_3 tend to 0 as $\varepsilon\to 0$ by using Minkowski's integral inequality in the same way as in the proof of [7, Lemma 6. 3].

Theorem 6. 2. *Let $1<p<\infty$. The operator P, defined by* (6. 1) *for all $f\in L^p(\Gamma\diagdown\mathfrak{N})$, is a bounded projection onto $H^p(\Gamma\diagdown D)$. The boundary function of Pf is given by*

$$(Pf)^\sim(g)=\tfrac{1}{2}f(g)+\lim_{\varepsilon\to 0}\int_{\Gamma\diagdown\mathfrak{N}} k^{*,\varepsilon}(g,h)\,f(h)\,dh. \tag{6. 3}$$

Proof. Pf is holomorphic since S^* is holomorphic. By Theorem 2. 2, the limit (6. 3) exists in $L^p(\Gamma\diagdown\mathfrak{N})$ and defines a bounded operator. Lemma 6. 1 shows that the boundary function of Pf is given by (6. 3). It follows that the L^p-norm of $(Pf)_t^\sim$ is bounded for small $t>0$; it is also bounded for large t since Pf, being holomorphic, is bounded at infinity. So, $Pf\in H^p(\Gamma\diagdown D)$.

To see that P is a projection, we have to show $P^2=P$. Now, $P^2f=Pf$ for $f\in L^2(\Gamma\diagdown\mathfrak{N})$ by Theorem 5. 2, and $L^2(\Gamma\diagdown\mathfrak{N})$ is dense in each $L^p(\Gamma\diagdown\mathfrak{N})$. To see that the range of P is all of $H^p(\Gamma\diagdown D)$, we note that $Pf=f$ for all $H^2(\Gamma\diagdown D)$ and $H^2(\Gamma\diagdown D)$ is dense in each $H^p(\Gamma\diagdown D)$ since $f=\lim_{t\to 0} f_t$ for all $f\in H^p(\Gamma\diagdown D)$, by Theorem 3. 1.

§ 7. Remarks on the Bergman kernel

Let $\mathscr{L}^2(\Gamma\diagdown D)$ be the space of holomorphic functions square integrable on $\Gamma\diagdown D$. A fundamental domain for the action of Γ on D is given by $(\operatorname{Re} z_1, z_2)\in\Omega$, $\operatorname{Im} z_1 > \frac{1}{2}|z_2|^2$ where Ω is as in § 4. Inner products in $\mathscr{L}^2(\Gamma\diagdown D)$ are computed by integrating on this domain.

It is easy to see that the system (4. 3) with the omission of φ_{00} is an orthogonal basis in $\mathscr{L}^2(\Gamma\diagdown D)$ and one has

$$\|\varphi_{ml}\|^2_{\mathscr{L}^2(\Gamma\diagdown D)} = \frac{\operatorname{Im}\tau}{2\pi m}\|\varphi_{ml}\|^2_{H^2(\Gamma\diagdown D)}. \tag{7. 1}$$

Denoting by K^* the Bergman kernel of $\Gamma\diagdown D$, we have by (4. 3), (4. 4),

$$K^*(z,w) = \frac{2^{1/2}\pi k}{(\operatorname{Im}\tau)^{5/2}}\sum_{\substack{m\equiv 0\ (\mathrm{mod}\, k)\\ m>0}} m^{3/2}\sum_{l=0}^{m-1}\varphi_{ml}(z)\overline{\varphi_{ml}(w)}. \tag{7. 2}$$

By (4. 3) and (5. 6), this is equal to $-2i\dfrac{\partial}{\partial z_1}S^*(z,w)$. Let K denote the Bergman kernel of D. From the explicit formulas in [6], we see that $K(z,w)=-2i\dfrac{\partial}{\partial z_1}S(z,w)$. Thus, by (5. 3), it follows that

$$K^*(z,w) = \sum_{\gamma\in\Gamma}K(\gamma z, w). \tag{7. 3}$$

References

[1] L. Auslander, L. Green et al., *Flows on homogeneous spaces,* Ann. of Math. Studies 53, Princeton University Press (Princeton, 1963).

[2] J. Brezin, Harmonic analysis on nilmanifolds, *Trans. Amer. Math. Soc.*, **150** (1970), 611—618.

[3] A. P. Calderón and A. Zygmund, Singular integrals and periodic functions, *Studia Math.*, **14** (1954), 249—271.

[4] P. Cartier, Quantum mechanical commutation relations and Theta functions, *Proceedings of Symposia in Pure Mathematics,* Amer. Math. Soc., **9** (1966), 361—383.

[5] F. Conforto, *Abelsche Funktionen und algebraische Geometrie*, Springer-Verlag (1956).

[6] A. Korányi, The Poisson integral for generalized half-planes and bounded symmetric domains, *Ann. of Math.*, **82** (1965), 332—350.

[7] A. Korányi and S. Vági, Singular integrals on homogeneous spaces and some problems of classical analysis, *Ann. Scuola Norm. Sup. Pisa, Classe di Scienze,* **25** (1971), 575—648.

[8] C. C. Moore, Decomposition of unitary representations defined by discrete subgroups of nilpotent groups, *Ann. of Math.*, **82** (1965), 146—182.

[9] I. I. Pjateckiĭ-Šapiro, *Geometry of classical domains and theory of automorphic functions* (Moscow, 1961) (in Russian).

[10] C. L. Siegel, *Analytic functions of several complex variables*. Lecture notes, The Institute of Advanced Study, Princeton 1948, reprinted 1962.
[11] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Clarendon Press (Oxford, 1937).
[12] A. Zygmund, *Trigonometric series*, Cambridge University Press (1959).

(Received June 6, 1972)